

Trajectory probability hypothesis density filter

Ángel F. García-Fernández, Lennart Svensson

Abstract—This paper presents the probability hypothesis density (PHD) filter for sets of trajectories. The resulting filter, which is referred to as trajectory probability density filter (TPHD), is capable of estimating trajectories in a principled way without requiring to evaluate all measurement-to-target association hypotheses. As the PHD filter, the TPHD filter is based on recursively obtaining the best Poisson approximation to the multitrajectory filtering density in the sense of minimising the Kullback-Leibler divergence. We also propose a Gaussian mixture implementation of the TPHD recursion, the Gaussian mixture TPHD (GMTPHD), and a computationally efficient implementation, the L -scan GMTPHD, which only updates the PDF of the trajectory states of the last L time steps.

Index Terms—Random finite sets, multitarget tracking, sets of trajectories, PHD.

I. INTRODUCTION

In multiple target tracking, we are primarily interested in estimating an unknown number of target trajectories, also known as tracks, from a sequence of measurements. The random finite set (RFS) framework and finite-set statistics (FISST) developed by Mahler make up a widely accepted framework for dealing with multiple object systems from a Bayesian point of view [1], [2].

In the RFS framework, the multitarget state at the current time is usually represented by a set that contains single target states. The multitarget filtering probability density function (PDF) of the current set of targets given all available measurements contains the information of interest about these targets. Theoretically, this multitarget PDF can be calculated recursively via the prediction and update steps but approximations are required in practice.

The first proposed filter within the RFS framework was the probability hypothesis density (PHD) filter [3]. The PHD filter has been employed in an extensive range of applications such as multitarget tracking [4], [5], robotics [6], [7], computer vision [8], road mapping [9] and sensor control [10], [11]. The PHD filter fits into the assumed density filtering (ADF) framework and propagates a Poisson (multitarget) PDF on the current set of targets through the prediction and update steps [12]. In a Poisson PDF, the cardinality of the set is Poisson distributed and its elements are independent and identically distributed (IID). If the output of either prediction or update step is no longer Poisson, the PHD filter obtains its best Poisson approximation by minimising the Kullback-Leibler divergence (KLD) from the true density to a Poisson density.

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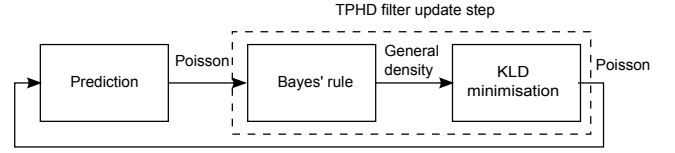


Figure 1: TPHD filter diagram. The TPHD filter assumes that the multitrajectory PDFs involved are Poisson (on the space of sets of trajectories). The output of Bayes' rule is no longer Poisson but, in order to be able to perform the Bayesian recursion, it obtains the best Poisson approximation to the filtering density by minimising the KLD.

The output of the update step is not Poisson in general so a KLD minimisation is required but, in the prediction, it is not necessary if target births are Poisson and there is no target spawning. The most popular implementations of the PHD filter are based on Gaussian mixtures and sequential Monte Carlo methods [5], [13], [14].

The main appealing characteristics of the PHD filter are its low computational burden and ease of implementation. It avoids the measurement-to-target association problem and we just need to calculate the PHD of the multitarget filtering PDF, which is defined over the single target space. It also has some drawbacks such as the spooky effect [2] or the fact that it does not build tracks, sequences of target state estimates that belong to the same target. The smoother version of the PHD filter [15], [16] does not avoid these problems. Despite the fact that the PHD filter is unable to provide tracks in a mathematically rigorous way, several ad-hoc track building procedures have been proposed [17]–[19]. In this paper we address this shortcoming of the PHD filter and develop a PHD filter that provides tracks from first principles: the trajectory PHD (TPHD) filter.

The TPHD filter follows the same scheme as the PHD filter with a fundamental difference, instead of using a set of targets as the state, it uses a set of trajectories. The theory for performing multiple target tracking using sets of trajectories is also based on Mahler's FISST framework and is explained in [20]. A set of trajectories is a variable that encapsulates all relevant information to build tracks, i.e., the number of trajectories, start times, lengths and sequence of target states for each trajectory. In the TPHD filter, we therefore propagate a Poisson (multitrajectory) PDF on the space of the set of trajectories through the prediction and update steps. We do not consider target spawning and assume Poisson target births so a KLD minimisation is only required after the update step [12]. A diagram of the resulting Bayesian recursion is given in Figure 1.

In this paper, we also propose an implementation of the TPHD filter based on Gaussian mixtures, which follows the

spirit of the Gaussian mixture PHD filter [13]. The resulting Gaussian mixture TPHPD (GMTPHD) filter builds trajectories without the use of labels under a Poisson approximation whose PHD is represented by a Gaussian mixture. Additionally, we propose a version of the GMTPHD filter with lower computational burden called the L -scan GMTPHD filter. In practice, this filter only updates the multitrajectory PDF of the trajectory states of the last L time instant leaving the rest unaltered, which is quite efficient for implementation. The theoretical foundation of the L -scan GMTPHD filter is also based on the ADF framework and KLD minimisations.

The rest of the paper is organised as follows. In Section II, we present some background material for sets of trajectories. Section III explains the Poisson RFS of trajectories and some of its properties. The TPHPD filter is derived in Section IV and its Gaussian mixture implementations in Section V. Simulation results are shown in Section VI. Finally, conclusions are given in Section VII.

II. BACKGROUND

In this section, we describe some background material on multiple target tracking using sets of trajectories [20]. In Section II-A, we introduce the variables we consider. We present the set integral and probability hypothesis density for sets of trajectories in Sections II-B and II-C, respectively.

A. Variables

A single target state $x \in D$, where $D = \mathbb{R}^{n_x}$, contains the information of interest about the target, e.g., its position and velocity. A set of single target states \mathbf{x} belongs to $\mathcal{F}(D)$ where $\mathcal{F}(D)$ denotes the set of all finite subsets of D . We are ultimately interested in estimating all target trajectories, where a trajectory consists of a sequence of target states that can start at any time step and end any time later on. Mathematically, we represent a trajectory as a variable $X \in T$, where $T = \mathbb{N} \times \uplus_{i=1}^{\infty} D^i$ and \uplus stands for disjoint union and it is used in this paper to highlight that the sets are disjoint, as in [4]. We can write $X = (t, x^{1:i})$ where t is the initial time of the trajectory, i is its length and $x^{1:i} = (x^1, \dots, x^i)$ denotes a sequence of length i that contains the target states at consecutive time steps of the trajectory. Similarly to the set \mathbf{x} of targets, we denote a set of trajectories as $\mathbf{X} \in \mathcal{F}(T)$.

Given a single target trajectory $X = (t, x^{1:i})$, the set $\tau^k(X)$ of the target state at time k is

$$\tau^k(X) = \begin{cases} \{x^{k+1-t}\} & t \leq k \leq t+i-1 \\ \emptyset & \text{elsewhere.} \end{cases}$$

As the trajectory exists from time step t to $t+i-1$, the set is empty if k is outside this interval. Given a set \mathbf{X} of trajectories, the set $\tau^k(\mathbf{X})$ of target states at time k is

$$\tau^k(\mathbf{X}) = \bigcup_{X \in \mathbf{X}} \tau^k(X). \quad (1)$$

In RFS filtering, it is assumed that two or more targets at a given time cannot have an identical state. The corresponding assumption for sets of trajectories is that any two trajectories $X, Y \in \mathbf{X}$ must satisfy that $\tau^k(X) \cap \tau^k(Y) = \emptyset$ for all $k \in \mathbb{N}$.

B. Set integral

Given a real-valued function $\pi(\cdot)$ on the set of trajectories space $\mathcal{F}(T)$, its set integral is [20]:

$$\begin{aligned} \int \pi(\mathbf{X}) \delta \mathbf{X} &= \sum_{n=0}^{\infty} \frac{1}{n!} \int \pi(\{X_1, \dots, X_n\}) dX_{1:n} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(t_{1:n}, i_{1:n}) \in \mathbb{N}^{2n}} \int \pi(\{(t_1, x_1^{1:i_1}), \dots, (t_n, x_n^{1:i_n})\}) \\ &\quad \times d(x_1^{1:i_1}, \dots, x_n^{1:i_n}) \end{aligned} \quad (2)$$

where $X_{1:n} = (X_1, \dots, X_n)$. The integral goes through all possible cardinalities, start times, lengths and target states of the trajectories. Function $\pi(\cdot)$ is a multitrajectory PDF if $\pi(\cdot) \geq 0$ and its set integral is one.

C. Probability hypothesis density

Given a multitrajectory PDF $\pi(\cdot)$, its PHD is [1, Eq. (16.33)]

$$\begin{aligned} D_\pi(X) &= \int \pi(\{X\} \cup \mathbf{X}) \delta \mathbf{X} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int \pi(\{X, X_1, \dots, X_n\}) dX_{1:n} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(t_{1:n}, i_{1:n}) \in \mathbb{N}^{2n}} \int \pi(\{X, (t_1, x_1^{1:i_1}), \dots, (t_n, x_n^{1:i_n})\}) \\ &\quad \times d(x_1^{1:i_1}, \dots, x_n^{1:i_n}). \end{aligned} \quad (3)$$

As in the PHD for RFS of targets, integrating the PHD in a region $A \subseteq T$ gives us the expected number of trajectories in this region [2, Eq. (4.76)]:

$$\hat{N}_A = \int_A D_\pi(X) dX \quad (5)$$

$$= \sum_{(t,i) \in \mathbb{N}^2} \int 1_A(t, x^{1:i}) D_\pi(t, x^{1:i}) dx^{1:i} \quad (6)$$

where $1_A(\cdot)$ is the indicator function of a subset A . Therefore, the expected number of trajectories is given by substituting $A = T$ in (5).

Example 1. Let us consider a multitrajectory PDF $\nu(\cdot)$ with

$$D_\nu(1, x^1) = \mathcal{N}(x^1; 10, 1) + \mathcal{N}(x^1; 1000, 1) \quad (7)$$

$$D_\nu(1, x^{1:2}) = \mathcal{N}(x^{1:2}; (10, 10.1), \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}), \quad (8)$$

where $\mathcal{N}(\cdot; m, P)$ is a Gaussian PDF with mean m and covariance matrix P , and zero otherwise. The expected number of trajectories that start at time one with length 1 is given by substituting $A = \{1\} \times D$ in (5) so

$$\hat{N}_A = \int D_\nu(1, x^1) dx^1 = 2$$

The expected number of trajectories is $\hat{N}_T = 3$. \square

III. POISSON TRAJECTORY RFS

In this section, we introduce the Poisson point process over the trajectory space [21], which we refer to as Poisson trajectory RFS, and some of its properties. Section III-A describes its multitrajectory PDF. How to sample from a Poisson multitrajectory PDF is detailed in Section III-B. A result on marginalisation for Poisson trajectory RFS is provided in Section III-C. A KLD minimisation theorem is given in Section III-D.

A. Probability density function

In the Poisson RFS, the cardinality of the set is Poisson distributed and its elements are independent and identically distributed (IID). A Poisson multitrajectory PDF $\nu(\cdot)$ has the form

$$\nu(\{X_1, \dots, X_n\}) = e^{-\lambda_\nu} \lambda_\nu^n \prod_{j=1}^n \check{\nu}(X_j) \quad (9)$$

where $\check{\nu}(\cdot)$ is a single trajectory PDF, which implies

$$\int \check{\nu}(X) dX = 1,$$

and $\lambda_\nu \geq 0$. A Poisson multitrajectory PDF is characterised by either its PHD $D_\nu(X) = \lambda_\nu \check{\nu}(X)$ or by λ_ν and $\check{\nu}(\cdot)$ [2]. As a result, using (5), the expected number of trajectories is $\hat{N}_T = \lambda_\nu$. Further, its cardinality distribution is given by [20]

$$\begin{aligned} \rho_\nu(n) &= \frac{1}{n!} \int \nu(\{X_1, \dots, X_n\}) dX_{1:n} \\ &= \frac{1}{n!} e^{-\lambda_\nu} \lambda_\nu^n \end{aligned} \quad (10)$$

Example 2. Let us consider the Poisson multitrajectory PDF of Example 1. Using (10), its cardinality distribution is Poisson with parameter $\lambda_\nu = 3$ and, therefore, its single trajectory PDF is

$$\check{\nu}(X) = \frac{1}{3} D_\nu(X)$$

where $D_\nu(\cdot)$ is given by (7) and (8). \square

B. Sampling

In this subsection, we indicate how to draw samples from a Poisson multitrajectory PDF. Given a single trajectory PDF $\check{\nu}(\cdot)$, the probability that the trajectory starts at time t and has duration i is

$$P_{\check{\nu}}(t, i) = \int \check{\nu}(t, x^{1:i}) dx^{1:i}. \quad (11)$$

That is, we integrate over all possible trajectories with start time t and duration i . Given the start time t and duration i , the PDF of the states is

$$\check{\nu}(x^{1:i}|t, i) = \frac{\check{\nu}(t, x^{1:i})}{P_{\check{\nu}}(t, i)}. \quad (12)$$

Therefore, the procedure to draw samples from a Poisson multitrajectory PDF is shown in Algorithm 1.

Algorithm 1 Sampling from a Poisson multitrajectory PDF

Input: Poisson PDF $\nu(\cdot)$.
Output: Sample $X \sim \nu(\cdot)$.
- Initialisation: $Y^0 = \emptyset$.
- Sample $n \sim \rho_\nu(\cdot)$, see (10).
for $j = 1$ to n **do**
- Sample $(t, i) \sim P_{\check{\nu}}(\cdot)$, see (11).
- Sample $x^{1:i} \sim \check{\nu}(\cdot|t, i)$, see (12).
- Set $Y^j = Y^{j-1} \cup \{(t, x^{1:i})\}$.
end for
- Set $X = Y^n$.

C. Marginalisation for Poisson multitrajectory PDF

Given a Poisson multitrajectory PDF $\nu(\cdot)$, the multitarget PDF $\nu_\tau^k(\cdot)$ of the set of targets at time k is Poisson with PHD

$$D_{\nu_\tau^k}(y) = \sum_{t=1}^k \sum_{j=0}^{\infty} \int \int D_\nu(t, x^{1:k-t}, y, z^{1:j}) dx^{1:k-t} dz^{1:j} \quad (13)$$

where $(t, x^{1:k-t}, y, z^{1:j})$ denotes a trajectory that starts at time t with states $(x^{1:k-t}, y, z^{1:j})$ so it has a duration $k - t + 1 + j$. This result is obtained from the basic properties of Poisson processes [21, Chap. 2].

Example 3. We consider the Poisson multitrajectory PDF of Example 1. Using (13), the set of targets at time 1 is Poisson distributed with PHD

$$D_{\nu_\tau^1}(y) = 2\mathcal{N}(y; 10, 1) + \mathcal{N}(y; 1000, 1).$$

The expected number of targets at time 1 is 3. \square

D. KLD minimisation

Using FISST [2], the KLD from $\pi(\cdot)$ to $\nu(\cdot)$ is given by

$$\begin{aligned} D(\pi \| \nu) &= \int \pi(\mathbf{X}) \log \frac{\pi(\mathbf{X})}{\nu(\mathbf{X})} \delta \mathbf{X} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(t_{1:n}, i_{1:n}) \in \mathbb{N}^{2n}} \\ &\quad \int \pi(\{(t_1, x_1^{1:i_1}), \dots, (t_n, x_n^{1:i_n})\}) \\ &\quad \times \log \frac{\pi(\{(t_1, x_1^{1:i_1}), \dots, (t_n, x_n^{1:i_n})\})}{\nu(\{(t_1, x_1^{1:i_1}), \dots, (t_n, x_n^{1:i_n})\})} \\ &\quad \times d(x_1^{1:i_1}, \dots, x_n^{1:i_n}). \end{aligned} \quad (14)$$

Then, we obtain the following theorem for Poisson RFS and KLD minimisation.

Theorem 4. Given a multitrajectory PDF $\pi(\cdot)$, the PHD that characterises Poisson multitrajectory PDF $\nu(\cdot)$ that minimises the KLD $D(\pi \| \nu)$ satisfies $D_\nu(\cdot) = D_\pi(\cdot)$.

Theorem 4 is proved in [3, Theorem 4] for RFS of targets. The extension for trajectories is provided in Appendix A.

IV. TRAJECTORY PHD FILTER

In this section, we derive the TPHD filter. In Section IV-A, we present the Bayesian filtering recursion for sets of trajectories. The prediction and update steps of the TPHD filter are given in Sections IV-B and IV-C, respectively. A discussion on marginalisation is provided in Section III-C.

A. Bayesian filtering recursion

The objective is to calculate the multitrajectory filtering PDF $\pi^k(\cdot)$ at time k , which is the multitrajectory PDF of the set of trajectories up to time step k conditioned on the measurements up to time step k . We assume that the set of trajectories at time k evolves with a transition density $f^k(\cdot|\cdot)$. In addition, given the targets at time k , the set \mathbf{z}^k of measurements at time k has a density $\ell^k(\cdot|\tau^k(\mathbf{X}))$. The multitrajectory filtering PDF $\pi^k(\cdot)$ at time k can then be calculated via the prediction and update steps [20]:

$$\omega^k(\mathbf{X}) = \int f^k(\mathbf{X}|\mathbf{Y}) \pi^{k-1}(\mathbf{Y}) \delta\mathbf{Y} \quad (15)$$

$$\pi^k(\mathbf{X}) = \frac{\ell^k(\mathbf{z}^k|\tau^k(\mathbf{X})) \omega^k(\mathbf{X})}{\ell^k(\mathbf{z}^k)} \quad (16)$$

where $\omega^k(\cdot)$ is the predicted multitrajectory PDF at time k , which denotes the PDF of the set of trajectories up to time step k given the measurements up to time step $k-1$, and

$$\ell^k(\mathbf{z}^k) = \int \ell^k(\mathbf{z}^k|\tau^k(\mathbf{X})) \omega^k(\mathbf{X}) \delta\mathbf{X}$$

is the density of the measurements.

B. Prediction

We make the following assumptions in the prediction step:

- P1 Targets move independently, survive with a probability $p_S(\cdot)$ and move with a transition PDF $g(\cdot|\cdot)$.
- P2 The set of new born targets has a Poisson multitarget density $\beta_\tau(\cdot)$ and is independent of the surviving targets.
- P3 The multitrajectory filtering PDF $\pi^{k-1}(\cdot)$ at time $k-1$ is Poisson.

Let $1_A(\cdot)$ denote the indicator function of set A and $\mathbb{N}_k = \{1, \dots, k\}$. Then, the relation between predicted PHD at time k and the PHD of the posterior at time $k-1$ is given by the following theorem.

Theorem 5 (Prediction). *Under Assumptions P1-P3, the predicted PHD $D_{\omega^k}(\cdot)$ at time k is*

$$D_{\omega^k}(X) = D_{\xi^k}(X) + D_{\beta^k}(X)$$

where

$$D_{\beta^k}(t, x^{1:i}) = D_{\beta_\tau}(x^1) 1_{\{k\}}(t) 1_{\{1\}}(i)$$

and

$$\begin{aligned} D_{\xi^k}(t, x^{1:i}) &= D_{\pi^{k-1}}(t, x^{1:i}) 1_{\mathbb{N}_{k-2}}(t+i-1) \\ &+ (1 - p_S(x^i)) D_{\pi^{k-1}}(t, x^{1:i}) 1_{\{k-1\}}(t+i-1) \\ &+ p_S(x^{i-1}) g(x^i|x^{i-1}) D_{\pi^{k-1}}(t, x^{1:i-1}) 1_{\{k\}}(t+i-1) \end{aligned}$$

if $t+i-1 \leq k$ or zero otherwise.

Theorem 5 is proved in Appendix B. The predicted PHD is the sum of the PHD $D_{\beta^k}(\cdot)$ of the trajectories born at time step k and the PHD $D_{\xi^k}(\cdot)$ of the trajectories present at previous time steps. The end time of trajectory $(t, x^{1:i})$ is $t+i-1$. Therefore, the three terms of $D_{\xi^k}(\cdot)$ have clear interpretations in terms of trajectory end times. The prediction step does not change the PHD for the trajectories that ended before time step k . The PHD of the trajectories that end at time step k is multiplied by $1 - p_S(x^i)$, which represents the probability of not surviving. For the surviving trajectories, we multiply the PHD by the transition PDF and the survival probability.

C. Update

We make the following assumptions in the update step [12]:

- U1 The set $\mathbf{z}^k = \mathbf{z}_1^k \cup \mathbf{z}_2^k$ where \mathbf{z}_1^k and \mathbf{z}_2^k are the independent RFSs of measurements coming from targets and clutter, respectively.
- U2 Given the set of targets $\mathbf{x} = \{x_1, \dots, x_n\}$ at time k , $\mathbf{z}_1^k = \bigcup_{i=1}^n \tilde{\mathbf{z}}_{1,i}$ where $\tilde{\mathbf{z}}_{1,i}$ $i = 1, \dots, n$ are independent and $\tilde{\mathbf{z}}_{1,i} = \emptyset$ with probability $1 - p_D(x_i)$ and $\tilde{\mathbf{z}}_{1,i} = \{z_i\}$ where z_i has a PDF $l(\cdot|x_i)$.
- U3 The set \mathbf{z}_2^k of clutter measurements is a Poisson process with multiobject PDF $c(\cdot)$.
- U4 The predicted multitrajectory PDF $\omega^k(\cdot)$ is Poisson.

Let Ξ_{n,n_z} denote the set that contains all the vectors $\sigma = (\sigma_1, \dots, \sigma_{n_z})$ that indicate associations of n_z measurements to n targets, which can be either detected or undetected. Under Assumption A2, we should note that only one measurement can be associated to a target. If $\sigma \in \Xi_{n,n_z}$, $\sigma_i = j \in \{1, \dots, n_z\}$ indicates measurement j is associated with target i and $\sigma_i = 0$ indicates that target i has not been detected. Under Assumptions U1, U2 and U3, which define the standard measurement model, the PDF of the measurement given the state is [2, Eq. (7.21)]

$$\begin{aligned} &\ell^k(\{z_1, \dots, z_{n_z}\} | \{x_1, \dots, x_n\}) \\ &= e^{-\lambda_c} \left[\prod_{i=1}^{n_z} \lambda_c \check{c}(z_i) \right] \left[\prod_{i=1}^n (1 - p_D(x_i)) \right] \\ &\quad \times \sum_{\sigma \in \Xi_{n,n_z}} \prod_{i:\sigma_i > 0} \frac{p_D(x_i) l(z_{\sigma_i}|x_i)}{(1 - p_D(x_i)) \lambda_c \check{c}(z_{\sigma_i})}. \end{aligned} \quad (17)$$

where λ_c and $\check{c}(\cdot)$ characterise $c(\cdot)$, see (9).

Let $L_{\mathbf{z}^k}(\cdot)$ denote the PHD filter pseudolikelihood function, which is given by [2, Sec. 8.4.3]

$$\begin{aligned} L_{\mathbf{z}^k}(x) &= 1 - p_D(x) + p_D(x) \\ &\quad \times \sum_{z \in \mathbf{z}^k} \frac{l(z|x)}{\lambda_c \check{c}(z) + \int p_D(y) l(z|y) D_{\omega_\tau^k}(y) dy} \end{aligned}$$

with $D_{\omega_\tau^k}(\cdot)$ representing the PHD of the targets at time k of density $\omega^k(\cdot)$, see (13):

$$D_{\omega_\tau^k}(y) = \sum_{t=1}^k \int D_{\omega^k}(t, x^{1:k-t}, y) dx^{1:k-t}.$$

Then, we prove in Appendix C the TPHD filter update step:

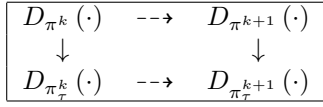


Figure 2: Relation between the PHDs $D_{\pi^k}(\cdot)$ and $D_{\pi^{k+1}}(\cdot)$, which are obtained using the PHD filter for targets, and the trajectory PHDs $D_{\pi^k}(\cdot)$ and $D_{\pi^{k+1}}(\cdot)$ of the RFS of trajectories. Dashed arrows represent the prediction and update steps and the solid arrows for marginalisation.

Theorem 6 (Update). *Under Assumptions U1-U4, the updated PHD $D_{\pi^k}(\cdot)$ at time k is*

$$D_{\pi^k}(t, x^{1:i}) = D_{\omega^k}(t, x^{1:i}) [1_{\mathbb{N}_{k-1}}(t+i-1) + 1_{\{k\}}(t+i-1) L_{\mathbf{z}^k}(x^i)]$$

if $t+i-1 \leq k$ or zero otherwise.

As in the prediction step, the update step does not change the PHD for the trajectories that have died before time step k . It should be noted that Bayes update (16) uses a likelihood (17) which involves a summation over all target to measurements associations in the multitarget space. In contrast, the TPHD filter update is similar to the PHD filter update in the sense that it uses a pseudolikelihood function $L_{\mathbf{z}^k}(\cdot)$ which is defined on the single target space and only involves associations between a single target and the measurements.

D. Marginalisation

It can be checked that if we perform marginalisation at time step k , see (13), and apply the (target) PHD prediction and update, we obtain the same result as if we apply the (trajectory) PHD prediction and update and then apply marginalisation. This is illustrated in Figure 2. Consequently, the information regarding the set of targets at the current time step is the same for the PHD and TPHD filters. For example, the estimated cardinality of alive trajectories/targets is the same for both filters.

V. GAUSSIAN MIXTURE TPHD FILTER

In this section we propose a Gaussian mixture implementation of the TPHD filter. The prediction and update steps are provided in Section V-A. We motivate why the GMTPHD should only be used to track alive trajectories in Section V-B. The L -scan GMTPHD, which is a computationally efficient implementation, is described in Section V-C. An estimation procedure for the GMTPHD filter is given in Section V-D.

A. Prediction and update

The recursion of the GMTPHD filter is quite similar to the GMPHD filter [13]. We use the notation

$$\mathcal{N}(t, x^{1:i}; t^k, m^k, P^k) = \begin{cases} \mathcal{N}(x^{1:i}; m^k, P^k) & i = i^k, t = t^k \\ 0 & \text{otherwise} \end{cases} \quad (18)$$

where $i^k = \dim(m^k)/n_x$. Equation (18) represents a single trajectory Gaussian PDF with start time t^k , duration i^k , mean $m^k \in \mathbb{R}^{i^k n_x}$ and covariance matrix $P^k \in \mathbb{R}^{i^k n_x \times i^k n_x}$

evaluated at $(t, x^{1:i})$. We also use \otimes to indicate Kronecker product, $0_{m,n}$ is a zero matrix of dimension $m \times n$ and superscript T stands for transpose.

We make the additional assumptions

- A1 The survival and detection probabilities, p_S and p_D , are constants that do not depend on the target state.
- A2 $g(x^i | x^{i-1}) = \mathcal{N}(x^i; Fx^{i-1}, Q)$.
- A3 $l(z|x) = \mathcal{N}(z; Hx, R)$.
- A4 The PHD of the birth density $\beta^k(\cdot)$ is a Gaussian mixture

$$D_{\beta^k}(X) = \sum_{j=1}^{J_{\beta}^k} w_{\beta,j}^k \mathcal{N}(X; k, m_{\beta,j}^k, P_{\beta,j}^k) \quad (19)$$

where $J_{\beta}^k \in \mathbb{N}$ is the number of components, $m_{\beta,j}^k \in \mathbb{R}^{n_x}$ and $P_{\beta,j}^k \in \mathbb{R}^{n_x \times n_x}$.

It should be noted that the models provided by A1-A3 could be time varying but we do not wish to introduce additional notation. Under Assumptions A1-A4, P1-P3 and U1-U4, we can calculate the output of the TPHD filter in closed form. The GMTPHD filter prediction and update steps are provided in the rest of this section.

Proposition 7 (Prediction). *We denote the PHD of $\pi^k(\cdot)$ by*

$$D_{\pi^k}(X) = D_{\pi^k_*}(X) + D_{\pi^k_{\circ}}(X)$$

where

$$D_{\pi^k_*}(X) = \sum_{j=1}^{J^k} w_j^k \mathcal{N}(X; t_j^k, m_j^k, P_j^k)$$

$$D_{\pi^k_{\circ}}(X) = \sum_{j=1}^{J_{\circ}^k} w_{\circ,j}^k \mathcal{N}(X; t_{\circ,j}^k, m_{\circ,j}^k, P_{\circ,j}^k)$$

represent the PHD of alive and dead trajectories, i.e., $t_j^k + i_j^k - 1 = k$ and $t_{\circ,j}^k + i_{\circ,j}^k - 1 < k$ with $i_j^k = \dim(m_j^k)/n_x$ and $i_{\circ,j}^k = \dim(m_{\circ,j}^k)/n_x$. Then, the PHD of $\omega^{k+1}(\cdot)$ is

$$D_{\omega^{k+1}}(X) = p_S \sum_{j=1}^{J^k} w_j^k \mathcal{N}(X; t_j^k, m_{\omega,j}^{k+1}, P_{\omega,j}^{k+1}) + (1 - p_S) D_{\pi^k_*}(X) + D_{\pi^k_{\circ}}(X) + D_{\beta^{k+1}}(X)$$

where

$$m_{\omega,j}^{k+1} = \left[(m_j^k)^T, (\dot{F}_j m_j^k)^T \right]^T$$

$$P_{\omega,j}^{k+1} = \begin{bmatrix} P_j^k & \dot{F}_j P_j^k \\ P_j^k \dot{F}_j^T & \dot{F}_j P_j^k \dot{F}_j^T + Q \end{bmatrix}$$

$$\dot{F}_j = [0_{1, i_j^k-1}, 1] \otimes F.$$

Proposition 7 can be proved using Theorem 5. The GMTPHD filter prediction is similar to the GMPHD filter prediction with the main differences is that previous states are not integrated out and there is information about dead trajectories.

Proposition 8 (Update). *We denote the PHD of $\omega^k(\cdot)$ by*

$$D_{\omega^k}(X) = D_{\omega^k_*}(X) + D_{\omega^k_{\circ}}(X)$$

where

$$D_{\omega_*^k}(X) = \sum_{j=1}^{J_\omega^k} w_{\omega,j}^k \mathcal{N}(X; t_{\omega,j}^k, m_{\omega,j}^k, P_{\omega,j}^k)$$

$$D_{\omega_\circ^k}(X) = \sum_{j=1}^{J_\circ^k} w_{\circ,j}^k \mathcal{N}(X; t_{\circ,j}^k, m_{\circ,j}^k, P_{\circ,j}^k)$$

represent the PHD of alive and dead trajectories. Then, PHD of $\pi^k(\cdot)$ is

$$D_{\pi^k}(X) = D_{\omega_\circ^k}(X) + (1 - p_D) D_{\omega_*^k}(X) + \sum_{z \in \mathbf{z}^k} \sum_{j=1}^{J^k} w_j(z) \mathcal{N}(X; t_{\omega,j}^k, m_j^k(z), P_j^k)$$

where

$$w_j(z) = \frac{p_D w_{\omega,j}^k q_j(z)}{\lambda_c \check{c}(z) + p_D \sum_{l=1}^{J_\omega^k} w_{\omega,l}^k q_l(z)}$$

$$q_j(z) = \mathcal{N}(z; \bar{z}_j, S_j)$$

$$\bar{z}_j = \dot{H}_j m_{\omega,j}^k$$

$$S_j = \dot{H}_j P_{\omega,j}^k \dot{H}_j^T + R$$

$$\dot{H}_j = [0_{1, i_{\omega,j}^k - 1}, 1] \otimes H$$

$$m_j^k(z) = m_{\omega,j}^k + P_{\omega,j}^k \dot{H}_j^T S_j^{-1} (z - \bar{z}_j)$$

$$P_j^k = P_{\omega,j}^k - P_{\omega,j}^k \dot{H}_j^T S_j^{-1} \dot{H}_j P_{\omega,j}^k.$$

where $i_{\omega,j}^k = \dim(m_{\omega,j}^k) / n_x$

Proposition 8 can be proved using Theorem 6. As $D_{\omega_*^k}(\cdot)$ and $D_{\omega_\circ^k}(\cdot)$ represent the alive and dead trajectories, respectively, it is met that $t_{\omega,j}^k + i_{\omega,j}^k - 1 = k$ and $t_{\circ,j}^k + i_{\circ,j}^k - 1 < k$. Also, the GMTPHD filter update is similar to the GMPHD filter update. The main differences is that we keep the PHD corresponding to dead trajectories and we update the whole trajectories. The updated weights of the alive components are the same as in the GMPHD filter because the likelihood only depends on the the current set of targets.

B. Tracking of only alive trajectories

In this section, we motivate why the practical GMTPHD implementations should not attempt to track the dead trajectories. As in the PHD filter, the Poisson approximation for the multitarget PDF of the current set of targets, is a strong approximation but yields acceptable results in many situations [2]. The Poisson approximation for the multitrajectory PDF of the current set of trajectories is even stronger due to the higher dimensionality of the state space. In practice, we argue that it is only useful to obtain information about the present trajectories at the current time step. The reason behind this is that, in the prediction step, those components which consider trajectories that have died before time step k have a weight that is multiplied by $(1 - p_S)$, see Proposition 7. The probability p_S of survival is usually close to one so these components have very low weights. In the next update and prediction steps, this weight remains unchanged due to the IID assumptions inherent in PHD filtering. As a result, all components that

represent dead trajectories have very low weight even if there were very likely in the past.

The conclusion is that the Poisson approximation to the full multitrajectory filtering PDF is not an accurate representation of the knowledge over all trajectories that have existed up to the current time. A related phenomenon occurs for particle filters [22]. Even though most particle filters approximate the posterior PDF over the whole trajectory, the marginal density at a time far away in the past is not properly represented due to the particle degeneracy [23]. Nevertheless, the TPHD filter provides a first order approximation of the PDF of the set of trajectories present at the current time step so we can use the TPHD filter recursion to approximate the posterior for the alive trajectories. The resulting pseudocodes for the prediction and update steps are provided by Algorithms 2 and 3, where we denote the parameters of the PHD mixtures as $\Phi_{\omega,j}^k = (w_{\omega,j}^k, t_{\omega,j}^k, m_{\omega,j}^k, P_{\omega,j}^k)$, $\Phi_j^k = (w_j^k, t_j^k, m_j^k, P_j^k)$ and $\Phi_{\beta,j}^k = (w_{\beta,j}^k, k, m_{\beta,j}^k, P_{\beta,j}^k)$.

Algorithm 2 Prediction step of the GMTPHD filter

Input: Updated PHD parameters $\{\Phi_j^k\}_{j=1}^{J^k}$.

Output: Predicted PHD parameters $\{\Phi_{\omega,j}^{k+1}\}_{j=1}^{J_\omega^{k+1}}$.

- Set $\Phi_{\omega,j}^{k+1} = \Phi_{\beta,j}^{k+1}$ for $j = \{1, \dots, J_\beta^{k+1}\}$. \triangleright New born targets
 - Set $p = J_\beta^{k+1} + 1$. $\triangleright p$ is an index on output mixture
 - for** $j = 1$ to J^k **do** \triangleright Alive trajectories
 - $w_{\omega,p}^{k+1} = p_S w_j^k$, $t_{\omega,p}^{k+1} = t_j^k$.
 - $m_{\omega,p}^{k+1} = \begin{bmatrix} (m_j^k)^T, (\dot{H}_j m_j^k)^T \end{bmatrix}^T$.
 - $P_{\omega,p}^{k+1} = \begin{bmatrix} P_j^k & \dot{H}_j P_j^k \\ P_j^k \dot{H}_j^T & \dot{H}_j P_j^k \dot{H}_j^T + Q \end{bmatrix}$.
 - $p = p + 1$.
 - end for**
 - $J_\omega^{k+1} = J_\beta^{k+1} + J^k$.
-

C. L-scan GMTPHD

In this section, we propose a computationally efficient implementation of the GMTPHD filter: the L-scan GMTPHD filter.

As the Gaussian mixture PHD filter, the Gaussian mixture TPHD filter has an increasing number of components as time progresses so we need to bound the number of components in practice. The simplest technique is to prune the components whose weight is below a threshold Γ_p and set a maximum number J_{max} of components [13]. In addition, if two components have a very similar current state, based on a Mahalanobis distance criterion, future measurements will affect both component weights and future states in a similar way. Therefore, we can remove components that are close to another component with higher weight. We account for this decrease in the number of components by increasing the weight of the component that has not been removed by the weights of the removed components. We refer to as this technique as absorption. The steps of the pruning and absorption algorithms for the GMTPHD are given in Algorithm 4.

Algorithm 3 Update step of the GMTPHD filter

Input: Predicted PHD parameters $\{\Phi_{\omega,j}^k\}_{j=1}^{J_\omega^k}$ and \mathbf{z}^k .

Output: Updated PHD parameters $\{\Phi_j^k\}_{j=1}^{J^k}$.

```
for  $j = 1$  to  $J_\omega^k$  do ▷ Preliminary moments
  -  $\bar{z}_j = \bar{H}_j m_{\omega,j}^k$ .
  -  $S_j = \bar{H}_j P_{\omega,j}^k \bar{H}_j^T + R$ .
  -  $K_j = P_{\omega,j}^k \bar{H}_j^T S_j^{-1}$ .
  -  $P_j = P_{\omega,j}^k - K_j \bar{H}_j P_{\omega,j}^k$ .
end for
- Set  $p = 1$ . ▷  $p$  is an index on output mixture
for  $j = 1$  to  $J_\omega^k$  do ▷ Non-detected components
  -  $\Phi_p^k = \Phi_{\omega,j}^{k|k-1}$ .
  -  $w_p^k = (1 - p_D) w_{\omega,j}^k$ .
  -  $p = p + 1$ .
end for
for each  $z \in \mathbf{z}^k$  do ▷ Detected components
  -  $s = 0$ . ▷ Sum of weights
  for  $j = 1$  to  $J_\omega^k$  do
    -  $w_p^k = p_D w_{\omega,j}^k \mathcal{N}(z; \bar{z}_j, S_j)$ .
    -  $m_p^k = m_{\omega,j}^k + K_j(z - \bar{z}_j)$ .
    -  $P_p^k = P_j$ .
    -  $s = s + w_p^k$ .
    -  $p = p + 1$ .
  end for
  for  $j = 1$  to  $J_\omega^k$  do
    -  $w_{p-j}^k = w_{p-j}^k / (\lambda_c \check{c}(z) + s)$ .
  end for
end for
-  $J^k = p - 1$ .
```

Algorithm 4 Pruning and absorption for the GMTPHD filter

Input: Posterior parameters $\{\Phi_j^k\}_{j=1}^{J^k}$, pruning threshold Γ_p , absorption threshold Γ_a , maximum number of terms J_{max} .

Output: Pruned posterior parameters $\{\Phi_{o,j}^k\}_{j=1}^{J^k}$.

```
- Set  $l = 0$  and  $I = \{j \in \{1, \dots, J^k\} : w_j^k > \Gamma_p\}$ .
while  $I \neq \emptyset$  do
  - Set  $l = l + 1$ .
  -  $j = \arg \max_{i \in I} w_i^k$ .
  -  $L = \left\{ i \in I : (\hat{m}_i^k - \hat{m}_j^k)^T (\hat{P}_j^k)^{-1} (\hat{m}_i^k - \hat{m}_j^k) < \Gamma_a \right\}$ 
  with  $\hat{m}_j^k \in \mathbb{R}^{n_x}$  and  $\hat{P}_j^k \in \mathbb{R}^{n_x \times n_x}$  denoting the mean and
  covariance matrix of the state at the current time step.
  -  $\Phi_{o,l}^k = \Phi_j^k$  with weight  $w_{o,l}^k = \sum_{i \in L} w_i^k$ .
  -  $I = I \setminus L$ .
end while
- If  $l > J_{max}$ , remove the  $l - J_{max}$  components with lowest weight
  from  $\Phi_{o,l}^k$ .
```

In addition, as time progresses, the lengths of the trajectories increase so, eventually, the direct implementation of the GMTPHD is not computationally feasible. Fortunately, in practice, measurements at the current time step only have a significant impact on the trajectory state estimates for recent time steps. Based on this insight combined with the ADF framework and KLD minimisation, we propose a computationally efficient, single trajectory L -scan filter in Appendix D. The PDF that this filter propagates is composed by the joint PDF of the states of the last L time steps and independent PDFs for the previous states. We apply this filter to each mixture component of the GMTPHD posterior and the resulting algorithm is referred to

as L -scan GMTPHD.

The L -scan GMTPHD can be implemented as the GMTPHD with a minor modification in the prediction step, where we discard the correlations of states that happened L time steps before the current time step. Given a predicted PHD $D_{\omega^k}(\cdot)$, see Algorithm 2, its L -scan version is

$$D_{\omega^k}^{(L)}(X) = \sum_{j=1}^{J_\omega^k} w_{\omega,j}^k \mathcal{N}(X; t_{\omega,j}^k, m_{\omega,j}^k, P_{\omega,j}^{k(L)}) \quad (20)$$

where $P_{\omega,j}^{k(L)} = \text{diag}(\tilde{P}_j^{k-L+1:k}, \tilde{P}_j^{k-L}, \tilde{P}_j^{k-L-1}, \dots, \tilde{P}_j^k)$ with $\tilde{P}_j^{k-L+1:k}$ with dimensions $L \cdot n_x \times L \cdot n_x$ and \tilde{P}_j^k with dimensions $n_x \times n_x$. Matrix $\tilde{P}_j^{k-L+1:k}$ represents the joint covariance of the L last time instants, obtained from $P_{\omega,j}^k$, and \tilde{P}_j^k represents the covariance matrix of the state at time t , obtained from $P_{\omega,j}^k$. Therefore, we have independent Gaussian PDFs to represent the states outside the L -scan window and a joint Gaussian PDF for the states in the L -scan window. The steps of the L -scan GMTPHD filter are summarised in Algorithm 5. We want to highlight that even though the update step can be implemented exactly as in Algorithm 3, we can make it more efficient by only considering the states in the L -scan window as the rest are unaffected by the measurements.

It should be noted that the estimated number of alive trajectories and the target states at the current time is not affected by the selection of L . This implies that the estimated number of alive trajectories is equal to the number of targets of the GMPHD filter and the estimated targets at the current time using both the GMPHD or GMTPHD are alike. Interestingly, the 1-scan GMTPHD filter builds tracks roughly in the same way as suggested in [19] for the GMPHD filter. Nevertheless, rather than using first principles to build trajectories, the algorithm in [19] was obtained in an ad-hoc way by labelling each Gaussian component of the GMPHD implementation.

Algorithm 5 L -scan GMTPHD filter steps

```
- Initialisation:  $D_{\pi^0}(\cdot) = 0$ :  $J^0 = 0$ .
for  $k = 1$  to final time step do
  - Prediction using Algorithm 2 with this modification:
    - After calculating  $P_{\omega,j}^k$ , represent it in the form of  $P_{\omega,j}^{k(L)}$ ,
    see (20), by discarding correlations for states outside  $L$ -scan
    window.
    - Update using Algorithm 3.
    - Estimation of the alive trajectories, see Section V-D.
    - Pruning/Absorption using Algorithm 4.
  end for
```

D. Estimation

An optimal estimator should be the minimum of a metric or a cost function. Metrics in the case of sets of targets are for example the optimal subpattern assignment (OSPA) metric [24] and the generalised OSPA (GOSPA) [25]. For computational simplicity in the PHD filter, rather than minimising the OSPA metric, a suboptimal estimator is usually used. We adapt the estimator for the GMPHD filter described in [2, Sec. 9.5.4.4] for sets of trajectories. First, the number of trajectories

Table I: Parameters of the simulation

Parameter	p_S	τ	q	p_D	σ^2	λ_c
Value	0.99	0.5	3.24	0.9	16	50

is estimated as

$$\hat{N}^k = \text{round} \left(\sum_{j=1}^{J^k} w_j^k \right). \quad (21)$$

Then, the estimated set of trajectories corresponds to $\left\{ (t_{l_1}^k, m_{l_1}^k), \dots, (t_{l_{\hat{N}^k}}^k, m_{l_{\hat{N}^k}}^k) \right\}$ where $\{l_1, \dots, l_{\hat{N}^k}\}$ are the indices of the components with highest weights.

There are several assumptions for this sub-optimal estimator. For instance, the number J^k of components must be higher than or equal to \hat{N}^k and, if $(t_j^k, m_j^k) = (t_i^k, m_i^k)$, then $j = i$ as, otherwise, there can be two alike estimated trajectories. This estimator does not work well if there is a component with weight higher than two because there are at least two targets in that region but only one is reported. Nevertheless, this estimator is commonly used in the GMPHD filter and has a low computational complexity so we use it for the GMTPHD filter as well.

VI. SIMULATIONS

We proceed to assess the performance of the L -scan TPHD filter by simulations. We consider a target state $x = [p_x, \dot{p}_x, p_y, \dot{p}_y]^T$ where $[p_x, p_y]^T$ is the position vector and $x = [\dot{p}_x, \dot{p}_y]^T$ is the velocity vector. All the units of the quantities in this section are given in the international system. The parameters of the single-target dynamic process are

$$F = I_2 \otimes \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}, \quad Q = qI_2 \otimes \begin{pmatrix} \tau^3/3 & \tau^2/2 \\ \tau^2/2 & \tau \end{pmatrix}$$

where τ is the sampling time and q is a parameter. The parameters of the measurement model are

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad R = \text{diag}([\sigma^2, \sigma^2])$$

where σ^2 is the variance of the measurement noise for each component and we measure the target positions.

The clutter intensity is $c(z) = \lambda_c \cdot u_A(z)$ where $u_A(z)$ is a uniform PDF in region $A = [0, 2000] \times [0, 2000]$ and λ_c is the average number of clutter measurements per scan. The birth process is characterised by $J_\beta^k = 3$, $w_{\beta,j}^k = 0.1$, $P_{\beta,j}^k = \text{diag}([100, 100, 100, 100])$ for $j \in \{1, 2, 3\}$ and $m_{\beta,1}^k = [85, 0, 140, 0]^T$, $m_{\beta,2}^k = [-5, 0, 220, 0]^T$ and $m_{\beta,3}^k = [7, 0, 50, 0]^T$. The rest of the parameters of the simulation are given in Table I.

We consider the set of trajectories shown in Figure 3 and 100 time steps. We have implemented the L -scan TPHD filter with $L \in \{1, 2, 5, 10\}$ to evaluate the impact of this parameter in estimation performance. We use a pruning threshold $\Gamma_p = 10^{-4}$, absorption threshold $\Gamma_a = 0.1$ and limit the number of components to 30. Two exemplar outputs of the 10-scan TPHD filter are shown in Figure 4. At each time step, the TPHD provides an estimate of the set of present trajectories at the current time. The start and end times of an estimated

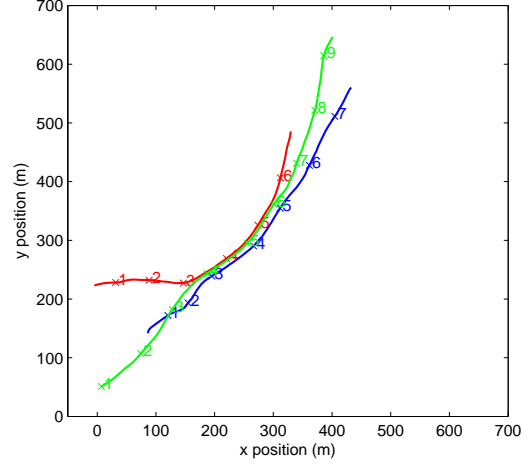


Figure 3: Scenario of the simulations. Target positions at time step $10k$ are marked by a cross and number k . Trajectories start at time steps $(1, 5, 10)$ and finish at time steps $(80, 70, 95)$.

trajectory do not depend on the choice of L so the output for any other L looks alike but with a different error.

In the following, we evaluate the performance of the L -scan TPHD filters by Monte Carlo simulation with 500 runs. We use a cost function based on OSPA metric for sets of targets [24]. At each time step, we take the set of alive trajectories and the estimated set of alive trajectories. Then, we compute the average OSPA metric for the corresponding set of targets at all previous time steps:

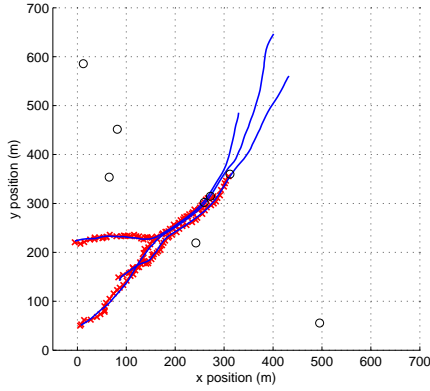
$$c(\mathbf{X}_a^k, \hat{\mathbf{X}}_a^k) = \frac{1}{k} \sum_{j=1}^k d(\tau^j(\mathbf{X}_a^k), \tau^j(\hat{\mathbf{X}}_a^k)) \quad (22)$$

where \mathbf{X}_a^k and $\hat{\mathbf{X}}_a^k$ represent the real set of alive trajectories at time k and its estimate, respectively, $d(\cdot, \cdot)$ is the OSPA metric with $p = 2$, Euclidean distance as base metric and $c = 10$.

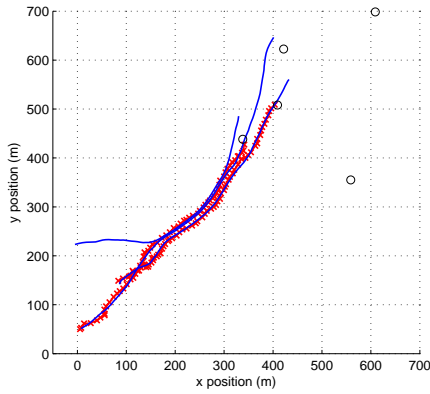
The resulting averaged OSPA errors for the L -scan TPHD filter are plotted in Figure 5. At the beginning, the filters have the same error but soon the differences start to appear. As expected, the error decreases as we increase L in the filter because we are considering a longer time window to update the trajectories. In addition, the running times of a non-optimised Matlab implementation of the filters on a Intel Core i7 laptop are basically the same for $L \in \{1, 2, 5, 10\}$: 7.7 seconds. In our implementation, the computational burden associated to operations resulting of using an L window of sizes between 1 and 10 is negligible compared to the computational burden of the rest of the filter. If we continue increasing L , the running time increases considerable, for example, 14.8 s for $L = 20$ and 28.0 s for $L = 30$.

The average cardinality of the estimated set of alive trajectories is shown in Figure 6. It does not change with L and is the same as for the GMPHD filter. The accuracy of the GMPHD filter cardinality estimates has been thoroughly analysed in [13]. The filter slightly underestimates the number of targets.

We also show the cost function (22) averaged over all time steps changing several parameters of the simulation in Table



(a) $k = 50$



(b) $k = 70$

Figure 4: Exemplar outputs of the TPHD at time steps 50 and 70. The blue lines represent the true trajectories, see Fig. 3. The red lines with crosses represent the estimated alive trajectories. Black circles represent the current measurements. The TPHD filter is able to estimate the alive trajectories.

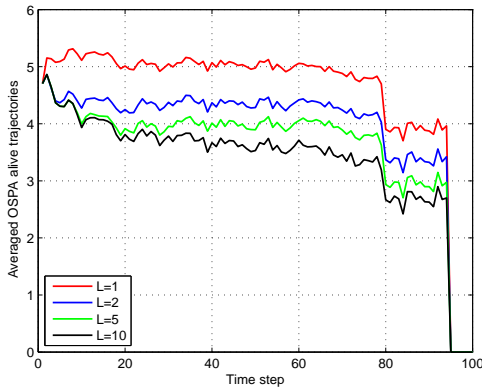


Figure 5: Averaged OSPA error for alive trajectories for the L -scan TPHD filter. Performance improves as we increase L .

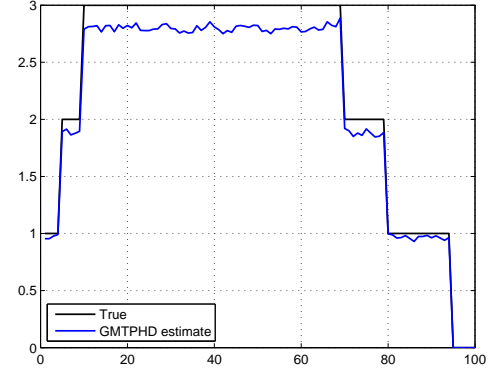


Figure 6: Average cardinality of the estimated set of alive trajectories. All L -scan GMTPHD filters provide the same estimate as the GMPHD filter.

Table II: Cost function averaged over all time steps

Changed parameter	L			
	1	2	5	10
No change	4.56	3.94	3.62	3.37
$\sigma^2 = 25$	5.15	4.49	4.06	3.78
$\sigma^2 = 9$	3.90	3.38	3.13	2.93
$\lambda_c = 70$	4.57	3.98	3.67	3.41
$\lambda_c = 90$	4.66	4.10	3.80	3.49
$p_D = 0.99$	3.69	2.96	2.50	2.47
$p_D = 0.95$	4.05	3.37	2.96	2.87
$p_D = 0.85$	5.30	4.85	4.62	4.21

II. Logically, with lower measurement noise or clutter rate or higher probability of detection, performance increases.

VII. CONCLUSIONS

In this paper we have presented the trajectory PHD filter and a Gaussian mixture implementation. The trajectory PHD filter fits in the context of assumed density filtering using sets of trajectories as state variable. It propagates a Poisson multitrajectory PDF through the filtering recursion and performs a KLD minimisation after each update step. Inspired by the PHD filter, the TPHD filter allows us to estimate the present trajectories at the current time in a computationally efficient way without arbitrary parameters such as labels.

The L -scan GMTPHD filter is an approximation of the GMTPHD with much lower computational burden. Its foundation is that measurements at the current time do not affect state estimates at time steps sufficiently far away in the past. This is achieved in a theoretically sound manner by resorting to the ADF framework with KLD minimisation. We use a single trajectory PDF that considers the last L time steps jointly and the previous states independently.

We have presented the GMTPHD filter for linear/Gaussian models. Nevertheless, as the GMPHD filter, the proposed algorithm can be adapted for nonlinear/non-Gaussian models using nonlinear Gaussian filters [26], [27].

APPENDIX A

In this appendix, we prove Theorem 4. A general RFS density $\pi(\cdot)$ can be written as [3]

$$\pi(\{X_1, \dots, X_n\}) = \rho_\pi(n) n! \pi_n(X_{1:n}) \quad (23)$$

where $\pi_n(\cdot)$ is a permutation invariant ordered PDF such that

$$\int \pi_n(X_{1:n}) dX_{1:n} = 1.$$

The marginal PDF of one trajectory of this density is

$$\begin{aligned} \tilde{\pi}_n(X) &= \int \pi_n(X, X_{2:n}) dX_{2:n} \\ &= \frac{1}{\rho_\pi(n) n!} \int \pi(\{X, X_2, \dots, X_n\}) dX_{2:n}. \end{aligned}$$

Substituting (9) into (14), we have that

$$\begin{aligned} D(\pi \parallel \nu) &= \int \pi(\mathbf{X}) \log \frac{\pi(\mathbf{X})}{\nu(\mathbf{X})} \delta \mathbf{X} \\ &= \sum_{n=0}^{\infty} \rho_\pi(n) \log \frac{\rho_\pi(n)}{e^{-\lambda_\nu} \lambda_\nu^n / n!} \\ &\quad + \sum_{n=0}^{\infty} \rho_\pi(n) \int \pi_n(X_{1:n}) \log \frac{\pi_n(X_{1:n})}{\prod_{j=1}^n \check{\nu}(X_j)} dX_{1:n}. \end{aligned} \quad (24)$$

We want to find λ_ν and $\check{\nu}(\cdot)$ that minimise (24). By derivating the first term w.r.t. λ_ν and equating it to zero, we obtain that the unique minimum is achieved by setting $\lambda_\nu = \sum_{n=0}^{\infty} n \rho_\pi(n)$. The second term can be written as

$$L[\check{\nu}] = - \int \sum_{n=0}^{\infty} \rho_\pi(n) n \tilde{\pi}_n(X) \log \check{\nu}(X) dX.$$

By KLD minimisation over single object spaces, we know that the previous functional is minimised by [28]

$$\begin{aligned} \check{\nu}(X) &= \frac{\sum_{n=0}^{\infty} \rho_\pi(n) n \tilde{\pi}_n(X)}{\sum_{n=0}^{\infty} \rho_\pi(n) n} \\ &= \frac{\sum_{n=0}^{\infty} \frac{1}{n!} \int \pi(\{X, X_1, \dots, X_n\}) dX_{1:n}}{\sum_{n=0}^{\infty} \rho_\pi(n) n} \\ &= \frac{D_\pi(X)}{\sum_{n=0}^{\infty} \rho_\pi(n) n} \end{aligned}$$

or equivalently, $D_\nu(\cdot) = D_\pi(\cdot)$.

APPENDIX B

In this appendix, we prove Theorem 5. A set of trajectories at time k can be decomposed as $\mathbf{W} \uplus \mathbf{X} \uplus \mathbf{Y} \uplus \mathbf{Z}$ where \mathbf{W} denotes the set of new born trajectories at time k , \mathbf{X} the set of trajectories present at times $k-1$ and k but not present at $k+1$, \mathbf{Y} the set of trajectories present at time $k-1$ but not present at time k and \mathbf{Z} the set of trajectories present at a time before $k-1$ but not at time k . We first clarify that if $(t, x^{1:i}) \in \mathbf{W}$, then, $t = k, i = 1$; if it belongs to \mathbf{X} , then $t < k, i = k-t+1$; if it belongs to \mathbf{Y} , then $t < k, i = k-t$; and finally, if it belongs to \mathbf{Z} , then, $t < k-1, i < k-t$. As $\mathbf{W}, \mathbf{X}, \mathbf{Y}$ and \mathbf{Z} are independent and Poisson distributed due to Assumptions P2-P3 so we can obtain their predicted PHDs independently. The overall predicted PHD is then the sum of these predicted PHDs due to the superposition of Poisson processes [12].

We use Theorem 5 in [20]. For dead trajectories, the prediction step leaves the multitrajectory PDF unaltered and so its PHD. The PHD of new born trajectories is analogous to the PHD of new born targets by setting the time to k and

duration to one. Using Theorem 5 in [20], we have that for $\mathbf{Y} = \{(t_1, x_1^{1:i_1}), \dots, (t_n, x_n^{1:i_n})\}$,

$$\begin{aligned} \omega^k(\mathbf{Y}) &= \pi^{k-1}(\{(t_1, x_1^{1:i_1}), \dots, (t_n, x_n^{1:i_n})\}) \\ &\quad \times \prod_{j=1}^n (1 - p_S(x_j^{i_n})). \end{aligned}$$

Using Assumption P3 and (9), we get that the predicted PHD, for $(t, x^{1:i}) \in \mathbf{Y}$, is

$$D_{\omega^k}(t, x^{1:i}) = (1 - p_S(x^i)) D_{\pi^{k-1}}(t, x^{1:i}).$$

Similarly, for $\mathbf{X} = \{(t_1, x_1^{1:i_1}), \dots, (t_n, x_n^{1:i_n})\}$,

$$\begin{aligned} \omega^k(\mathbf{X}) &= \pi^{k-1}(\{(t_1, x_1^{1:i_1-1}), \dots, (t_n, x_n^{1:i_n-1})\}) \\ &\quad \times \prod_{j=1}^n \left(g(x_j^{i_j} | x_j^{i_j-1}) p_S(x_j^{i_j-1}) \right) \end{aligned}$$

which implies that the predicted PHD for \mathbf{X} is the one indicated in Theorem 5 and we finish the proof.

APPENDIX C

In this appendix, we prove Theorem 6. As with the PHD filter, we first compute the density of the measurement at the current time [12].

A. Density of the measurement

Using (13) and Assumption U4, the multitarget predicted PDF at time k is Poisson with PHD

$$D_{\omega_\tau^k}(y) = \sum_{t=1}^k \int D_{\omega^k}(t, x^{1:k-t}, y) dx^{1:k-t} \quad (25)$$

where we have used the fact that $\omega^k(\cdot)$ is zero for trajectories present later than time k .

As in the usual PHD filter, due to the Poisson prior, the PDF of the measurement is Poisson with PHD [12]

$$D_{\ell^k}(z) = \int p_D(y) l(z|y) D_{\omega_\tau^k}(y) dy. \quad (26)$$

Therefore, the PDF of the measurement is

$$\begin{aligned} \ell^k(\mathbf{z}^k) &= e^{-\int p_D(y) l(z|y) D_{\omega_\tau^k}(y) dy - \lambda_c} \\ &\quad \prod_{z \in \mathbf{z}^k} [\lambda_c \check{c}(z) + p_D(y) l(z|y) D_{\omega_\tau^k}(y) dy]. \end{aligned} \quad (27)$$

B. Rest of the proof

Using (3), we calculate the updated PHD

$$\begin{aligned} D_{\pi^k}(X) &= \int \pi^k(\{X\} \cup \mathbf{X}) \delta \mathbf{X} \\ &= \frac{1}{\ell^k(\mathbf{z}^k)} \int \ell^k(\mathbf{z}^k | \tau^k(\{X\} \cup \mathbf{X})) \omega^k(\{X\} \cup \mathbf{X}) \delta \mathbf{X} \\ &= \frac{\lambda_{\omega^k} \check{\omega}^k(X)}{\ell^k(\mathbf{z}^k)} \int \ell^k(\mathbf{z}^k | \tau^k(X) \cup \tau^k(\mathbf{X})) \omega^k(\mathbf{X}) \delta \mathbf{X}. \end{aligned}$$

Given the set $\{x, x_1, \dots, x_n\}$ of targets at the current time, the PDF of the measurement given the current set of target decomposes as [20]

$$\begin{aligned} \ell^k(\mathbf{z}^k | \{x, x_1, \dots, x_n\}) \\ = (1 - p_D(x)) \ell^k(\mathbf{z}^k | \{x_1, \dots, x_n\}) \\ + p_D(x) \sum_{z \in \mathbf{z}^k} l(z|x) \ell^k(\mathbf{z}^k \setminus \{z\} | \{x_1, \dots, x_n\}) \end{aligned} \quad (28)$$

where $B \setminus A = \{z \in B | z \notin A\}$. Now, we consider two cases: X is not present at time k and X is present at time k .

1) *Case 1:* $\tau^k(X) = \emptyset$: In this case, $\ell^k(\mathbf{z}^k | \tau^k(X) \cup \tau^k(\mathbf{X})) = \ell^k(\mathbf{z}^k | \tau^k(\mathbf{X}))$ so

$$\begin{aligned} D_{\pi^k}(X) &= \lambda_{\omega^k} \check{\omega}^k(X) \\ &= D_{\omega^k}(X) \end{aligned}$$

2) *Case 2:* $\tau^k(X) \neq \emptyset$: In this case, we apply decomposition (28) and get

$$\begin{aligned} \ell^k(\mathbf{z}^k | \tau^k(X) \cup \tau^k(\mathbf{X})) \\ = (1 - p_D(\tau^k(X))) \ell^k(\mathbf{z}^k | \tau^k(\mathbf{X})) \\ + p_D(\tau^k(X)) \sum_{z \in \mathbf{z}^k} l(z | \tau^k(X)) \ell^k(\mathbf{z}^k \setminus \{z\} | \tau^k(\mathbf{X})) \end{aligned}$$

Following the same steps as in (target) PHD filter derivation [12], we get that

$$\begin{aligned} D_{\pi^k}(X) &= (1 - p_D(\tau^k(X))) \lambda_{\omega^k} \check{\omega}^k(X) \\ &+ p_D(\tau^k(X)) \lambda_{\omega^k} \check{\omega}^k(X) \\ &\times \sum_{z \in \mathbf{z}^k} \frac{l(z | \tau^k(X))}{\lambda_c \check{c}(z) + \int p_D(y) l(z|y) D_{\omega_T^k}(y) dy}. \end{aligned}$$

Finally, the PHD for cases 1 and 2 can be written as in Theorem 6.

APPENDIX D

In this appendix, we derive an L -scan single trajectory filter that jointly updates the PDF over the last L time steps and leaves unaltered the PDF at previous time steps. We use the ADF so we assume the posterior at time k is of a certain form and then we perform KLD minimisations to continue with the filtering recursion. For the sake of notational simplicity, we assume the trajectory exists at all time steps so we represent a trajectory as $x^{1:k}$. Let the posterior at time k be

$$\pi^k(x^{1:k}) = p^k(x^{k-L+1:k}) \prod_{i=1}^{k-L} q^i(x^i) \quad (29)$$

where $q^i(\cdot)$ is a PDF for the state at time step $i < k-L$ and $p^k(\cdot)$ is the joint PDF for the last L time steps. That is, the states corresponding to the last L time steps are considered jointly and the previous states are independent. This is an efficient representation that scales well for increasing time.

After a prediction and an update step on PDF (29), we obtain

$$\begin{aligned} \pi^{k+1}(x^{1:k+1}) &= r^{k+1}(x^{k-L+1:k+1}) \prod_{i=1}^{k-L} q^i(x^i) \\ r^{k+1}(x^{k-L+1:k+1}) &\propto l(z^{k+1}|x^{k+1}) g(x^{k+1}|x^k) \\ &\times p^k(x^{k-L+1:k}) \end{aligned}$$

where $l(z^{k+1}|\cdot)$ and $g(\cdot|\cdot)$ represent the likelihood and the dynamic model respectively. Projecting back to a PDF of the form (29), such that

$$\pi^{k+1}(x^{1:k+1}) = p^{k+1}(x^{k-L+2:k+1}) \prod_{i=1}^{k+1-L} q^i(x^i),$$

and minimising the KLD $D(\pi^{k+1} || \pi^{k+1})$ [28], we have to perform marginalisation such that

$$\begin{aligned} p^{k+1}(x^{k-L+2:k+1}) &= \int r^{k+1}(x^{k-L+1:k+1}) dx^{k-L+1} \\ q^{k+1-L}(x^{k+1-L}) &= \int r^{k+1}(x^{k-L+1:k+1}) dx^{k-L+2:k+1}. \end{aligned}$$

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